On Symplectically Harmonic Forms on Six-dimensional Nilmanifolds

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Abstract

In the present paper we study the variation of the dimensions h_k of spaces of symplectically harmonic cohomology classes (in the sense of Brylinski) on closed symplectic manifolds. We give a description of such variation for all 6-dimensional nilmanifolds equipped with symplectic forms. In particular, it turns out that certain 6-dimensional nilmanifolds possess families of homogeneous symplectic forms ω_t for which numbers $h_k(M, \omega_t)$ vary with respect to t. This gives an affirmative answer to a question raised by Boris Khesin and Dusa McDuff. Our result is in contrast with the case of 4-dimensional nilmanifolds which do not admit such variations by a remark of Dong Yan.

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1 Introduction

Given a symplectic manifold (M^{2m}, ω), we denote by $[\omega] \in H^2(M)$ the de Rham cohomology class of ω. Here the notation M^{2m} means that M is a $2m$-dimensional manifold. Furthermore, we denote by $L_\omega : \Omega^k(M) \to \Omega^{k+2}(M)$ the multiplication by ω and by $L_{[\omega]} : H^k(M) \to H^{k+2}(M)$ the induced homomorphism in the de Rham cohomology $H^*(M)$ of M. As usual, we write L instead of L_ω or $L_{[\omega]}$ if there is no danger of confusion. We say that a symplectic manifold (M^{2m}, ω) satisfies the *Hard Lefschetz condition* if, for every k, the homomorphism

$$L^k : H^{m-k}(M) \to H^{m+k}(M)$$

is surjective. In view of the Poincaré duality, for closed manifolds M it means that every L^k is an isomorphism.

In 1988 J. L. Brylinski [2] introduced the concept of symplectically harmonic forms (resp. Poisson harmonic forms), defined for any symplectic manifold (resp. Poisson manifold). Further he conjectured that on compact symplectic manifolds, every de Rham cohomology class has a symplectically harmonic representative. In fact, this conjecture asks about the possibility of constructing of a symplectic Hodge theory.

Brylinski proved that this conjecture is true for compact Kähler manifolds. However, it is not true in general, as it was shown in [5, 9, 15]. Mathieu [9] proved the following theorem, which applies to an arbitrary (not necessarily compact) symplectic manifold.

Theorem 1.1 *A symplectic manifold (M^{2m}, ω) satisfies the Brylinski conjecture if and only if it satisfies the Hard Lefschetz condition. In other words, the following assertions are equivalent:*

(i) *every de Rham cohomology class has a symplectically harmonic representative;*

(ii) *for every $k \le m$, the homomorphism $L^k : H^{m-k}(M) \longrightarrow H^{m+k}(M)$ is surjective.*

Mathieu's proof involves the representation theory of quivers and Lie superalgebras. An alternative and nice proof can be found in the paper of Yan [15], who studies a special type of infinite dimensional $\mathfrak{sl}(2)$-representation and, basing on this, proves a duality theorem for symplectically harmonic forms which, in turn, implies Theorem 1.1.

Given a symplectic manifold (M^{2m}, ω), let $\Omega^k_{\text{hr}}(M) = \Omega^k_{\text{hr}}(M, \omega)$ denote the subspace of symplectically harmonic forms of $\Omega^k(M)$. The form α is called symplectically harmonic if $d\alpha = 0 = \delta\alpha$, where $\delta = (-1)^{k+1} * d *$ and $*$ is the symplectic star operator, see Section 2. We set

$$H^k_{\text{hr}}(M) = H^k_{\text{hr}}(M, \omega) := \Omega^k_{\text{hr}}(M)/(\operatorname{Im} d \cap \Omega^k_{\text{hr}}(M))$$

and

$$h_k(M) = h_k(M, \omega) := \dim H^k_{\text{hr}}(M, \omega).$$

Since every symplectically harmonic form is closed, $H^k_{\rm hr}(M)$ is a subgroup of $H^k(M)$ and $h_k \le b_k$.

Of course, the above definition of the numbers h_k is not symmetric: one can also consider the "dual" numbers

$$h^*_k(M) = h^*_k(M,\omega) := \dim\left(\Omega^k_{\rm hr}/\operatorname{Im}\delta \cap \Omega^k_{\rm hr}\right).$$

It turns out that the duality isomorphism $* : \Omega^{m-k}_{\rm hr} \to \Omega^{m+k}_{\rm hr}$ yields the equality $h_{m-k} = h^*_{m+k}$, see Section 7.

Relating with the study of symplectically harmonic forms, we are interested in the following question raised by Boris Khesin and Dusa McDuff, see Yan [15].

Question: Which compact manifolds M possess a continuous family ω_t of symplectic forms such that $h_k(M,\omega_t)$ varies with respect to t?

This question, according to Khesin, is probably related to group theoretical hydrodynamics and geometry of diffeomorphism groups. Some indirect indications for an existence of such relations can be found in [1].

Since we want to consider manifolds as in the question, it makes sense to give them a certain name. So, let us call a closed smooth manifold M *flexible* if M possesses a continuous family of symplectic forms ω_t, $t \in [a,b]$, such that $h_k(M,\omega_a) \ne h_k(M,\omega_b)$ for some k. So, the above question asks about the existence of flexible manifolds.

Yan [15] has studied the case of closed 4-manifolds. He proved that 4-dimensional nilmanifolds are not flexible. He has also found examples of flexible 4-manifolds. Actually, one step in this proof of the existence is wrong, but the whole proof can easily be repaired, and hence the existence result holds. See Section 4 for details.

So, passing to higher dimensions, we have the following question:

Question: Do there exist flexible nilmanifolds of dimension ≥ 6?

Related with this question, we show that some 6-dimensional nilmanifolds are flexible.

Some words about tools. First, in order to prove the flexibility, we must be able to compute the symplectically harmonic Betti numbers. It turns out that, for every closed symplectic manifold (M^{2m},ω) and $k = 1$ or 2 we have

$$h_{2m-k}(M,\omega) = \operatorname{rank}\left(L^{m-k}_{[\omega]} : H^k(M) \to H^{2m-k}(M)\right),$$

see 3.2 and 3.4. So, a purely cohomological information is enough in order to compute h_{2m-k}. Furthermore, if $h_{2m-k}(M,\omega_1) \ne h_{2m-k}(M,\omega_2)$, $k = 1,2$, then M is flexible, see 2.11 and 3.2.

In general, not only symplectically harmonic forms but also Poisson harmonic forms are being of great interest in different areas of mathematics and physics, [3].

We use the sign $\boxed{QED}$ in order to indicate the end of a proof. However, if we formulate a claim without proof, we put the sign $\square$ in the end of the claim.

2 Symplectically harmonic forms

Let (M^{2m}, ω) be a symplectic manifold. It is well known that there exists a unique non-degenerate Poisson structure Π associated with the symplectic structure (see, for example, [8, 14]), that is, Π is a skew-symmetric tensor field of order 2 such that $[\Pi, \Pi] = 0$, where $[-, -]$ is the Schouten-Nijenhuis bracket.

The Koszul differential $\delta : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ is defined for symplectic manifolds, and more generally, for Poisson manifolds as

$$\delta = [i(\Pi), d].$$

Brylinski have proved in [2] that the Koszul differential is a symplectic codifferential of the exterior differential, with respect to the symplectic star operator. We choose the volume form associated to the symplectic form, that is, $v_M = \omega^m/m!$. Then we define the symplectic star operator

$$* : \Omega^k(M) \longrightarrow \Omega^{2m-k}(M),$$

by the condition $\beta \wedge (*\alpha) = \Lambda^k(\Pi)(\beta, \alpha)v_M$, for all $\alpha, \beta \in \Omega^k(M)$. The symplectic star operator satisfies the identities

$$*^2 = id, \quad \delta = (-1)^{k+1} * d*, \quad \text{and} \quad i(\Pi) = L^* := - * L * .$$

Furthermore, if M is a symplectic manifold M then the operators L, d, δ and L^* (acting on the algebra $\Omega^*(M)$) satisfy the following commutator relations:

$$[L, d] = 0, \quad [L, \delta] = -d, \quad [L^*, \delta] = 0, \quad [L^*, d] = -\delta. \tag{2.1}$$

Remark 2.1 The symplectic star operator was first considered by Libermann [8] as $*(\alpha) = i(\mu^{-1}\alpha)v_M$, where μ is the canonical isomorphism between the exterior algebras of vector fields and forms. She has also introduced and studied the operators $\delta = (-1)^{k+1} {*d*}$ and $L^* = - * L*$. In particular, she has proved (using the Lepage decomposition) that

$$\{\alpha \in \Omega^{m-k}(M) \mid L^{k+1}\alpha = 0\} = \{\alpha \in \Omega^{m-k}(M) \mid L^*\alpha = 0\}. \tag{2.2}$$

(See [15] for a proof using the theory of $\mathfrak{sl}(2)$-representations.)

Definition 2.2 A k-form α on the symplectic manifold M is called *symplectically harmonic* if $d\alpha = \delta\alpha = 0$. We denote the space of harmonic k-forms by $\Omega^k_{\mathrm{hr}}(M)$.

It is clear from (2.1) that the form $\omega \wedge \alpha$ is symplectically harmonic whenever α is. Hence, for every k we have the mapping $L : \Omega^k_{\mathrm{hr}}(M) \longrightarrow \Omega^{k+2}_{\mathrm{hr}}(M)$.

We set

$$H^k_{\rm hr}(M)=\Omega^k_{\rm hr}(M)/{\rm Im}\, d\cap \Omega^k_{\rm hr}(M) \text{ and } h_k=h_k(M,\omega)=\dim H^k_{\rm hr}(M).$$

So, for every symplectic manifold M, its de Rham cohomology $H^*(M)$ contains a symplectically harmonic subspace $H^*_{\rm hr}(M)$. We say that a de Rham cohomology class is *symplectically harmonic* if it contains a symplectically harmonic representative, i.e. if it belongs to the image of the inclusion $H^*_{\rm hr}(M)\subset H^*(M)$. Finally, we say that a manifold M is *flexible* if M possesses a continuous family of symplectic forms ω_t, $t\in[a,b]$, such that $h_k(M,\omega_a)\neq h_k(M,\omega_b)$ for some k.

Remark 2.3 In the general case of a (degenerate) Poisson manifold (M,Π), we say that a k-form α is *Poisson harmonic* if $d\alpha=0=\delta\alpha$. Notice that for a Poisson manifold, in particular for a symplectic manifold, $\Delta=d\delta+\delta d\equiv 0$, contrarily to the Riemannian case.

Remark 2.4 Recall that a *symplectomorphism* between two symplectic manifolds (M,ω_1) and (N,ω_2) is a diffeomorphism $\phi: M\longrightarrow N$ such that $\phi^{\#}(\omega_2)=\omega_1$. It is easy to see that

$$\phi^*(H^k_{\rm hr}(N))=H^k_{\rm hr}(M).$$

where $\phi^*: H^k(N)\to H^k(M)$ is the induced homomorphism in the de Rham cohomology. In other words, $H^*_{\rm hr}(-)$ is a symplectic invariant. In particular, if $h_k(M,\omega_1)\neq h_k(M,\omega_2)$ for two symplectic forms ω_1, ω_2 on M then ω_1 and ω_2 are not symplectomorphic.

We do not know whether a smooth map (not a diffeomorphism) ψ with $\psi^{\#}\omega_2=\omega_1$ induces a map of symplectically harmonic cohomology.

Proposition 2.5 ([15]) *For every symplectic manifold (M,ω), the homomorphism*

$$L^k:\Omega^{m-k}_{\rm hr}(M)\longrightarrow \Omega^{m+k}_{\rm hr}(M)$$

is an isomorphism. □

Corollary 2.6 *The homomorphism*

$$L^k: H^{m-k}_{\rm hr}(M)\longrightarrow H^{m+k}_{\rm hr}(M)$$

is an epimorphism. In particular, $h_{m-k}\geq h_{m+k}$. □

Corollary 2.7 *Let (M^{2m},ω) be a symplectic manifold. Then*

$$H^{m+k}_{\rm hr}(M)={\rm Im}\{L^k: H^{m-k}_{\rm hr}(M)\to H^{m+k}(M)\}\subset H^{m+k}(M).$$

Proof : It is a direct consequence of the commutativity of the diagram

$$\begin{array}{ccc} \Omega^{m-k}_{\rm hr}(M) & \xrightarrow{L^k} & \Omega^{m+k}_{\rm hr}(M) \\ \Big\downarrow & & \Big\downarrow \\ H^{m-k}_{\rm hr}(M) & \xrightarrow{L^k} & H^{m+k}_{\rm hr}(M) \end{array} \tag{2.3}$$

since the top map L^k is an isomorphism by 2.5 and both vertical maps are the epimorphisms.

$\boxed{QED}$

Corollary 2.8 *Let (M^{2m}, ω) be a closed symplectic manifold. If $h_{m+k}(M) = b_{m+k}(M)$ then $h_{m-k}(M) = b_{m-k}(M)$.* □

Proof : Because of 2.6 and Poincaré duality, $h_{m-k} \geq h_{m+k} = b_{m+k} = b_{m-k} \geq h_{m-k}$.

$\boxed{QED}$

Corollary 2.9 *Let (M, ω) be a symplectic manifold. If $b_k(M) = 0$ for some $k \leq m$, then $h_{2m-i}(M, \omega) = 0$ for $i \leq k$ with $k - i$ even.*

Proof : It follows from 2.7, since the homomorphism

$$L^{m-i} : H^i(M) \longrightarrow H^{2m-i}(M)$$

passes through the trivial group $H^k(M)$. $\boxed{QED}$

Lemma 2.10 *Let $\mathbb{L}$ be the space of all linear maps $\mathbb{R}^k \to \mathbb{R}^l$. Then the following holds:*

(i) *for every r the set*

$$\{A \in \mathbb{L} \mid \operatorname{rank} A \leq r\}$$

is an algebraic subset of $\mathbb{L}$. (Here we regard $\mathbb{L}$ as the space $\mathbb{R}^{kl}$ of $l \times k$-matrices whose entries are regarded as the coordinates);

(ii) *for every $m \leq \min\{k, l\}$ the set $\{A \in \mathbb{L} \mid \operatorname{rank} A \geq m\}$ is open and dense in $\mathbb{L}$;*

(iii) *let $A, B \in \mathbb{L}$ be two linear maps such that $\operatorname{rank} A < \operatorname{rank} B$. Then the set*

$$\Lambda = \{\lambda \in \mathbb{R} \mid \operatorname{rank}(A + \lambda B) \geq \operatorname{rank} B\}$$

is an open and dense subset of $\mathbb{R}$.

Proof : (i) This claim follows, because the rank of a matrix is equal to the order of the largest non-zero minor.

(ii) This claim follows from (i).

(iii) By (i), the set $\mathbb{R} \setminus \Lambda$ is an algebraic subset of $\mathbb{R}$. So, it suffices to prove that $\Lambda \neq \emptyset$. But, by (ii), $\operatorname{rank}(B + \mu A) > \operatorname{rank} A$ for μ small enough, and so $\Lambda \neq \emptyset$. $\boxed{QED}$

Corollary 2.11 *Let ω_0 and ω_1 be two symplectic forms on a manifold M^{2m}. Suppose that, for some $k > 0$, $h_{m-k}(\omega_0) = h_{m-k}(\omega_1)$, but $h_{m+k}(\omega_0) < h_{m+k}(\omega_1)$. Then, for every $\varepsilon > 0$, there exists $\lambda \in (0, \varepsilon)$ such that $h_{m+k}(\omega_0 + \lambda\omega_1) > h_{m+k}(\omega_0)$. Moreover, M is flexible provided that it is closed.*

Proof : The existence of λ follows from 2.7 and 2.10(iii). The flexibility of M follows, since $\omega_0 + t\omega_1$ is a symplectic form for t small enough. Indeed, if we set $\omega_t = \omega_0 + t\omega_1, t \in [0, \lambda]$, then $h_{m+k}(\omega_0) < h_{m+k}(\omega_\lambda)$. $\boxed{QED}$

Corollary 2.12 *Let (M^{2m}, ω_0) be a closed symplectic manifold. Given k with $0 < k < m$, suppose that $h_{m-k}(M, \omega) = b_{m-k}(M)$ for every symplectic form ω on M. Furthermore, suppose that there exists $x \in H^2(M)$ such that*

$$\operatorname{rank}\{L_x^k : H^{m-k}(M) \to H^{m+k}(M)\} > h_{m+k}(M, \omega_0).$$

Then M is flexible.

Proof : Take a closed 2-form α which represents x. Then $\omega_0 + t\alpha$ is a symplectic form for t small enough. Furthermore, by 2.7 and 2.10(iii), there exists arbitrary small λ such that $h_{m+k}(\omega_0 + \lambda\alpha) > h_{m+k}(\omega_0)$. Now the result follows from 2.11. $\boxed{QED}$

We set
$$\Omega_{\text{sympl}}(M) = \{\omega \in \Omega^2(M) \mid \omega \text{ is a symplectic form on } M\}$$
and define $\Omega(b, k) = \{\omega \in \Omega_{\text{sympl}} \mid h_k(M, \omega) = b\}$.

Corollary 2.13 *Let M^{2m} be a manifold that admits a symplectic structure. Suppose that, for some $k > 0$, $h_{m-k}(M, \omega)$ does not depend on the symplectic structure ω on M. Then the following three conditions are equivalent:*

(i) *the set $\Omega(b, m+k)$ is open and dense in $\Omega_{\text{sympl}}(M)$;*

(ii) *the interior of the set $\Omega(b, m+k)$ in $\Omega_{\text{sympl}}(M)$ is non-empty;*

(iii) *the set $\Omega(b, m+k)$ is non-empty and $h_{m+k}(M, \omega) \leq b$ for every $\omega \in \Omega_{\text{sympl}}(M)$.*

Proof : (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (iii). Suppose that there exists ω_0 with $h_{m+k}(M, \omega_0) > b$. Take ω in the interior of $\Omega(b, m+k)$. Then, in view of 2.10, there exists an arbitrary small λ such that $h_{m+k}(\omega + \lambda\omega_0) > b$, i.e. ω does not belong to the interior of $\Omega(b, m+k)$. This is a contradiction.

(iii) $\Rightarrow$ (i). This is true because of 2.10(ii). $\boxed{QED}$

So, the family $\{\Omega(b, m+k) | b = 0, 1, \ldots\}$ gives us a stratification of $\Omega_{\text{sympl}}(M)$ where the maximal strat is open and dense.

Lemma 2.14 *Let (M^{2m}, ω) be a closed symplectic manifold, and set*

$$\rho_{2k+1} = \operatorname{rank}\left\{L^{m-2k-1} : H^{2k+1}(M) \longrightarrow H^{2m-2k-1}(M)\right\}, \quad k = 0, \ldots, \left[\frac{m-1}{2}\right].$$

Then ρ_{2k+1} is an even number. Furthermore, $h_{2m-2k-1} \le \rho_{2k+1} \le b_{2k+1}$, and $\rho_{2k+1} = b_{2k+1}$ if and only if $L^{m-2k-1} : H^{2k+1}(M) \longrightarrow H^{2m-2k-1}(M)$ is surjective.

Proof : Let $p : H^{2k+1}(M) \otimes H^{2m-2k-1}(M) \longrightarrow \mathbb{R}$ be the usual non-singular pairing given by

$$p([\alpha], [\gamma]) = \int_M \alpha \wedge \gamma,$$

for $[\alpha] \in H^{2k+1}(M)$ and $[\gamma] \in H^{2m-2k-1}(M)$. Define a skew-symmetric bilinear form $\langle -, - \rangle : H^{2k+1}(M) \otimes H^{2k+1}(M) \longrightarrow \mathbb{R}$ via the formula

$$\langle [\alpha], [\beta] \rangle = p\left([\alpha], L^{m-2k-1}[\beta]\right),$$

for $[\alpha], [\beta] \in H^{2k+1}(M)$. It is easy to see that the rank of $\langle -, - \rangle$, which must be an even number $2l$ with $0 \le 2l \le b_{2k+1}$, is equal to ρ_{2k+1}.

The inequality $h_{2m-2k-1} \le \rho_{2k+1}$ follows from 2.7.

The last claim holds since, by the Poincaré duality, $b_{2k+1}(M) = b_{2m-2k-1}(M)$. $\boxed{QED}$

Remark 2.15 Lemma 2.14 yields the following well-known fact: if M in 2.14 satisfies the Hard Lefschetz condition, then all odd-dimensional Betti numbers are even.

Corollary 2.16 *Let (M^{2m}, ω) be a closed symplectic manifold. If $b_{2k+1}(M)$ is odd then $h_{2m-2k-1} < b_{2m-2k-1} = b_{2k+1}$. In particular, if $b_{2k+1} = 1$ then $h_{2m-2k-1} = 0$.* $\square$

Corollary 2.17 *If $b_{4k+2}(M) = 1$ then ρ_{2k+1} does not depend on symplectic structure on M.*

Proof : Because of the Poincaré duality, $b_{2m-4k-2} = 1$. So, $[\omega]^{m-2k-1}$, and hence

$$L^{m-2k-1} : H^{2k+1}(M) \longrightarrow H^{2m-2k-1}(M),$$

is determined uniquely up to non-zero multiplicative constant. $\boxed{QED}$

3 The numbers h_k and h_{2m-k} for k small

Proposition 3.1 *Let (M,ω) be a symplectic manifold, and let k be a non-negative integer number such that the following holds:*

(i) $L^{k+2} : H^{m-k-2}(M) \longrightarrow H^{m+k+2}(M)$ *is surjective;*

(ii) *If $a \in H^{m-k-2}(M)$ is not symplectically harmonic, then $La = 0$.*

Then every cohomology class in $H^{m-k}(M)$ is symplectically harmonic.

Proof : Here we use some ideas from [15]. It follows from (i) that

$$H^{m-k}(M) = \operatorname{Im} L + P_{m-k},$$

where $P_{m-k} = \{a \in H^{m-k}(M) \mid L^{k+1}a = 0\}$. Indeed, if $a \in H^{m-k}(M)$ then there exists $b \in H^{m-k-2}(M)$ such that $L^{k+1}a = L^{k+2}b$. Therefore, $a - b \wedge [\omega] \in P_{m-k}$, and

$$a = b \wedge [\omega] + (a - b \wedge [\omega]) \in \operatorname{Im} L + P_{m-k}.$$

Because of (ii), every class in $\operatorname{Im} L$ is symplectically harmonic. So, it suffices to prove that any cohomology class $a \in P_{m-k}$ is symplectically harmonic.

Let $a = [\alpha] \in P_{m-k}$ with $\alpha \in \Omega^{m-k}(M)$ closed. Since $L^{k+1}a = 0 \in H^{m+k+2}(M)$, there exists $\beta \in \Omega^{m+k+1}(M)$ such that $\alpha \wedge \omega^{k+1} = d\beta$. It is known [8] that $L^{k+1} : \Omega^{m-k-1}(M) \longrightarrow \Omega^{m+k+1}(M)$ is surjective. So, there exists $\gamma \in \Omega^{m-k-1}(M)$ with $\beta = \gamma \wedge \omega^{k+1}$, and hence $(\alpha - d\gamma) \wedge \omega^{k+1} = 0$. So, if we take $\overline{\alpha} = \alpha - d\gamma$, then $[\overline{\alpha}] = a$ and $L^{k+1}\overline{\alpha} = 0$. But the equality $L^{k+1}\overline{\alpha} = 0$ implies that $L^*\overline{\alpha} = 0$ in view of (2.2). Thus, $\overline{\alpha}$ is symplectically harmonic by (2.1). $\boxed{QED}$

Corollary 3.2 ([15]) *Let (M,ω) be an arbitrary symplectic manifold. Then every cohomology class in $H^k(M)$, $k = 0, 1, 2$, is symplectically harmonic.* □

Recall that a symplectic manifold (M^{2m},ω) is called a *manifold of Lefschetz type*, if the map

$$L : H^1(M) \longrightarrow H^{2m-1}(M)$$

is surjective. Notice that, similarly to 2.15, $b_1(M)$ is even for every closed manifold M of Lefschetz type.

Corollary 3.3 *Let (M^{2m},ω) be a manifold of Lefschetz type. Then every cohomology class in $H^3(M)$ is symplectically harmonic.* □

Corollary 3.4 *For every symplectic manifold* (M^{2m}, ω) *and* $k = 0, 1, 2,$

$$H^{2m-k}_{\mathrm{hr}}(M) = \mathrm{Im}\{L^{m-k} : H^k(M) \to H^{2m-k}(M)\} \subset H^{2m-k}(M).$$

Furthermore, if M *is a Lefschetz type manifold then*

$$H^{2m-3}_{\mathrm{hr}}(M) = \mathrm{Im}\{L^{m-3} : H^3(M) \to H^{2m-3}(M)\} \subset H^{2m-3}(M).$$

Proof : This follows from 2.7 because of 3.2 and 3.3. $\boxed{QED}$

Corollary 3.5 *For every closed symplectic manifold* (M^{2m}, ω) *the number* $h_{2m-1}(M^{2m}, \omega)$ *is even. Furthermore, if* $b_2 = 1$ *then* $h_{2m-1}(M^{2m}, \omega)$ *does not depend on* ω*.*

Proof : It follows from 2.14 and 2.17 since, by 3.4, $\rho_1 = h_{2m-1}$. $\boxed{QED}$

Corollary 3.6 *If* (M^{2m}, ω) *is a symplectic manifold that is not a manifold of Lefschetz type, then* $h_{2m-1} \le b_1 - 1$*. In particular, if* $b_1 = 2$*, then* $h_{2m-1} = 0$*.* $\square$

This is the case for compact non-toral nilmanifolds (see [14] and the table of the classification of 6-dimensional compact nilmanifolds in Section 5).

4 Yan's result on flexibility in dimension 4

According to Yan [15], 4-dimensional compact nilmanifolds are not flexible. Indeed, by 3.2, only h_3 may vary, but it turns out that h_3 is constant. Namely, based on certain results from [4], Yan [15] noticed the following relation for closed 4-dimensional nilmanifolds:

(i) if $b_1(M) = 2$, then $h_3 = 0$,

(ii) if $b_1(M) = 3$ (therefore, $b_2(M) = 4$), then $h_3 = 2$,

(iii) if $b_1(M) = 4$ (i.e. $M = \mathbb{T}^4$), then $h_3 = 4$.

On the other hand, Yan [15] has found closed 4-dimensional flexible manifolds, although his arguments need a certain correction (see below). Namely, he formulated without proof the following proposition, where M is assumed to be a closed 4-dimensional manifold.

Proposition 4.1 ([15, Prop. 4.1]) *The following assertions are equivalent:*

(i) *There exists a family ω_t of symplectic forms such that h_3 varies.*

(ii) *There exist two symplectic forms ω_1 and ω_2 such that $\operatorname{Im} L_{[\omega_1]} \neq \operatorname{Im} L_{[\omega_2]}$, where $L_{[\omega_i]}$ is the Lefschetz map respect to $[\omega_i]$, $(i = 1, 2)$.*

(iii) *There exists a symplectic form ω on M and a class $a \in H^2(M)$ such that $\operatorname{Im} L_a \not\subset \operatorname{Im} L_{[\omega]}$.*

(iv) *There exists a symplectic form ω on M such that $\operatorname{Im} L_{[\omega]}$ is not equal to the image of the cup product pairing $H^1(M) \otimes H^2(M) \longrightarrow H^3(M)$.* □

Concerning to this proposition, it is true that (i)⇒ (ii) ⇒ (iii) ⇒ (iv), but the Kodaira–Thurston manifold satisfies (iv) and does not satisfy (i). The Kodaira–Thurston manifold is obtained by taking the product of the Heisenberg manifold and the circle (this manifold is a compact nilmanifold). Its Sullivan minimal model has the form

$$(\Lambda(x_1, x_2, x_3, x_4), d) \quad \text{with} \quad dx_1 = dx_2 = dx_4 = 0,\ dx_3 = x_1 x_2,$$

where $\deg x_i = 1$ for $i = 1, 2, 3, 4$ and the generators x_1, x_2, x_3 come from the Heisenberg manifold. The cohomology class of the symplectic form is then given by the element $\omega = x_1 x_4 + x_2 x_3$. Now, from a direct computation we obtain that $\operatorname{Im} L_{[\omega]}$ is generated by the non-zero classes of $x_1 x_2 x_3$ and $x_2 x_3 x_4$ and the image of the cup product $H^1(M) \otimes H^2(M) \longrightarrow H^3(M)$ is generated by $x_1 x_2 x_3$, $x_1 x_3 x_4$ and $x_2 x_3 x_4$, so condition (iv) is satisfied. Moreover, it is easy to see that the Kodaira–Thurston manifold satisfies the condition (ii).

But, because of what we have said in the beginning of the section, any 4-dimensional nilmanifold (and hence the Kodaira-Thurston manifold) does not satisfy condition (i).

Yan's construction of flexible closed 4-dimensional manifolds is based on the following proposition.

Proposition 4.2 [15, Cor. 4.2] *Let (M^4, ω) be a closed symplectic manifold which satisfies the following conditions:*

(i) *the homomorphism $L_{[\omega]} : H^1(M) \longrightarrow H^3(M)$ is trivial;*

(ii) *the cup product $H^1(M) \otimes H^2(M) \longrightarrow H^3(M)$ is non-trivial.*

Then M is flexible. □

Yan regards this proposition as a corollary of his Proposition 4.1. As we have seen, the last one is wrong. However, Proposition 4.2 is correct because it is a special case of our Corollary 2.12.

Finally, Gompf [7, Observation 7] proved the existence of 4-manifolds as in 4.2.

5 Symplectically harmonic forms in homogeneous spaces

From the previous section we know that none 4-dimensional nilmanifold is flexible. The goal of this section is to demonstrate the existence of 6-dimensional flexible nilmanifolds.

A compact nilmanifold is a homogeneous space of the form G/Γ, where G is a simply connected nilpotent Lie group and Γ is a discrete co-compact subgroup of G, i.e. a lattice (co-compact means that G/Γ is compact). Recall that Γ is determined by G uniquely up to an isomorphism. In greater detail, if Γ and Γ' are two lattices in G then there exists an automorphism $\varphi : G \to G$ with $\varphi(\Gamma) = \Gamma'$, see [11]. Moreover, Γ determines G uniquely up to an isomorphism. In particular, the compact nilmanifold G/Γ determines and is completely determined by G.

Three important facts in the study of compact nilmanifolds are (see [14]):

(i) Let $\mathfrak{g}$ be a nilpotent Lie algebra with structural constants c_k^{ij} with respect to some basis, and let $\{\alpha_1, \dots, \alpha_n\}$ be the dual basis of $\mathfrak{g}^*$. Then in the Chevalley–Eilenberg complex $(\Lambda^*\mathfrak{g}^*, d)$ we have

$$d\alpha_k = \sum_{1 \le i < j < k} c_k^{ij} \alpha_i \wedge \alpha_j. \tag{5.1}$$

(ii) Let $\mathfrak{g}$ be the Lie algebra of a simply connected nilpotent Lie group G. Then, by Malcev's theorem, G admits a lattice if and only if $\mathfrak{g}$ admits a basis such that all the structural constants are rational.

(iii) By Nomizu's theorem, the Chevalley–Eilenberg complex $(\Lambda^*\mathfrak{g}^*, d)$ of $\mathfrak{g}$ is quasi-isomorphic to the de Rham complex of G/Γ. In particular,

$$H^*(G/\Gamma) \cong H^*(\Lambda^*\mathfrak{g}^*, d) \tag{5.2}$$

and any cohomology class $[a] \in H^k(G/\Gamma)$ contains a homogeneous representative α. Here we call the form α homogeneous if the pullback of α to G is left invariant.

Theorem 5.1 *There exist at least five* 6*-dimensional flexible nilmanifolds.*

To prove this theorem we run our fingers over all 34 6-dimensional compact nilmanifolds. The results are contained in the table below. The proofs take all the remained part of the section.

It follows from Corollary 3.2 that $\dim H^k_{\mathrm{hr}}(M) = \dim H^k(M)$, for $k \neq 3, 4, 5$. Therefore, we should study the behaviour of $h_k = \dim H^k_{\mathrm{hr}}(M)$ for degrees $k = 3, 4, 5$. We were not able to compute h_3, but we have found 5 manifolds with h_4 and/or h_5 varying. So, by

2.11, there are at least five 6-dimensional flexible nilmanifolds. (There are some reasons to conjecture that $h_3 = b_3$ for all closed 6-dimensional manifolds. So, if it is true then we have exactly five 6-dimensional flexible nilmanifolds.)

It turns out that every 6-dimensional real nilpotent Lie algebra admits a basis with rational structural constants. So, by what we said above, the 6-dimensional compact nilmanifolds are in a bijective correspondence with the 6-dimensional simply connected nilpotent Lie groups, and hence with the 6-dimensional nilpotent Lie algebras.

We use the classification of nilpotent Lie algebras given by Salamon [12]. It is based on the Morozov classification of 6-dimensional nilpotent Lie algebras [10]. We have added to Salamon's classification the symplectically harmonic Betti numbers $h_k(M)$ for $k = 4, 5$.

In the table Lie algebras appear lexicographically with respect to $(b_1, b_2, 6 - s)$ where s is the step length. The first two columns contain the Betti numbers b_1 and b_2 (notice that $b_3 = 2(b_2 - b_1 + 1)$ because of the vanishing of the Euler characteristic). The next column contains $6 - s$, where s is the step length.

The fourth column contains the description of the structure of the Lie algebra by means of the expressions of the form (5.1) in the Chevalley-Eilenberg complex. In view of 5.2, it means that, say, for the compact nilmanifold M from the second row, there exists a basis $\{\alpha_i\}_{i=1}^6$ of homogeneous 1-forms on M such that

$$d\alpha_1 = 0 = d\alpha_2,\ d\alpha_3 = \alpha_1 \wedge \alpha_2,\ d\alpha_4 = \alpha_1 \wedge \alpha_3,\ d\alpha_5 = \alpha_1 \wedge \alpha_4,\ d\alpha_6 = \alpha_3 \wedge \alpha_4 + \alpha_5 \wedge \alpha_2.$$

The column headed $\oplus$ indicates the dimensions of the irreducible subalgebras in case $\mathfrak{g}$ is not itself irreducible.

The next columns show the dimensions h_k for $k = 4, 5$. So, the column, say, h_4 contains all possible values of $h_4(M, \omega)$ which appear when ω runs over all symplectic forms on M. The sign "–" at a certain row means that the corresponding Lie algebra (as well as the compact nilmanifold) does not admit a symplectic structure.

For completeness, in the last columns we list the real dimension $\dim_{\mathbb{R}} \mathcal{S}(\mathfrak{g})$ of the moduli space of symplectic structures.

Convention 5.2 (i) From now on we write $\alpha_{ij\cdots k}$ instead of $\alpha_i \wedge \alpha_j \wedge \cdots \wedge \alpha_k$.

(ii) In future we say that a compact nilmanifold G/Γ has type, say (0,0,12,13,14,15) if the corresponding Lie algebra has the structure (0,0,12,13,14,15) (i.e. in our case, sits in the third row).

Six-dimensional real nilpotent Lie algebras

b_1	b_2	$6-s$	Structure	$\oplus$	h_4	h_5	$\dim_{\mathbb{R}}\mathcal{S}(\mathfrak{g})$
2	2	1	$(0,0,12,13,14+23,34+52)$		$-$	$-$	$-$
2	2	1	$(0,0,12,13,14,34+52)$		$-$	$-$	$-$
2	3	1	$(0,0,12,13,14,15)$		3	0	7
2	3	1	$(0,0,12,13,14+23,24+15)$		2	0	7
2	3	1	$(0,0,12,13,14,23+15)$		2	0	7
2	4	2	$(0,0,12,13,23,14)$		4	0	8
2	4	2	$(0,0,12,13,23,14-25)$		$2,3,4$	0	8
2	4	2	$(0,0,12,13,23,14+25)$		4	0	8
3	4	2	$(0,0,0,12,14-23,15+34)$		2	0	7
3	5	2	$(0,0,0,12,14,15+23)$		4	2	8
3	5	2	$(0,0,0,12,14,15+23+24)$		$3,4$	$0,2$	8
3	5	2	$(0,0,0,12,14,15+24)$	$1+5$	4	2	8
3	5	2	$(0,0,0,12,14,15)$	$1+5$	4	2	8
3	5	3	$(0,0,0,12,13,14+35)$		$-$	$-$	$-$
3	5	3	$(0,0,0,12,23,14+35)$		$-$	$-$	$-$
3	5	3	$(0,0,0,12,23,14-35)$		$-$	$-$	$-$
3	5	3	$(0,0,0,12,14,24)$	$1+5$	$-$	$-$	$-$
3	5	3	$(0,0,0,12,13+42,14+23)$		3	0	8
3	5	3	$(0,0,0,12,14,13+42)$		3	0	8
3	5	3	$(0,0,0,12,13+14,24)$		$2,3$	0	8
3	6	3	$(0,0,0,12,13,14+23)$		3	0	9
3	6	3	$(0,0,0,12,13,24)$		5	0	9
3	6	3	$(0,0,0,12,13,14)$		4	0	9
3	8	4	$(0,0,0,12,13,23)$		$7,8$	0	9
4	6	3	$(0,0,0,0,12,15+34)$		$-$	$-$	$-$
4	7	3	$(0,0,0,0,12,15)$	$1+1+4$	3	2	9
4	7	3	$(0,0,0,0,12,14+25)$	$1+5$	3	2	9
4	8	4	$(0,0,0,0,13+42,14+23)$		7	2	10
4	8	4	$(0,0,0,0,12,14+23)$		6	2	10
4	8	4	$(0,0,0,0,12,34)$	$3+3$	7	2	10
4	9	4	$(0,0,0,0,12,13)$	$1+5$	$7,8$	2	11
5	9	4	$(0,0,0,0,0,12+34)$	$1+5$	$-$	$-$	$-$
5	11	4	$(0,0,0,0,0,12)$	$1+1+1+3$	9	4	12
6	15	5	$(0,0,0,0,0,0)$	$1+\cdots+1$	15	6	15

Proof of Theorem 5.1

We prove the theorem via considering case by case. Namely, we study in more detail the cases which are proclaimed to be flexible. In view of 2.11, they are precisely the cases of compact nilmanifolds with varying symplectically harmonic Betti numbers h_k. Here the main tool for computing h_k is Corollary 3.4. In order to find symplectic structures on M, we use the following proposition.

Proposition 5.3 *Let M^{2n} be a compact manifold of the form G/Γ where Γ is a discrete subgroup of a Lie group G, and let $\omega \in \Omega^2(M)$ be a closed homogeneous 2-form such that $[\omega]^n \neq 0$. Then ω is a symplectic form on M.*

Proof : Since $[\omega]^n \neq 0$, we conclude that the linear form $\omega | T_x M$ is non-degenerate for some point $x \in M$. So, ω^n is non-degenerate since it is homogeneous. Thus, ω is non-degenerate.

$\boxed{QED}$

Proposition 5.4 *The compact nilmanifold M of the type $(0, 0, 12, 13, 23, 14 - 25)$ is flexible.*

Proof : According to our assumption about the type of M, there exists a basis $\{\alpha_i\}_{i=1}^6$ of homogeneous 1-forms on M such that

$$d\alpha_1 = d\alpha_2 = 0, \quad d\alpha_3 = \alpha_{12}, \quad d\alpha_4 = \alpha_{13}, \quad d\alpha_5 = \alpha_{23}, \quad d\alpha_6 = \alpha_{14} - \alpha_{25}.$$

Since the de Rham cohomology of the nilmanifold is isomorphic to the Chevalley-Eilenberg cohomology of the Lie algebra, we conclude that

$$\begin{aligned} H^1(M) &= \{[\alpha_1], [\alpha_2]\}, \\ H^2(M) &= \{[\alpha_{14}], [\alpha_{15} + \alpha_{24}], [\alpha_{26} - \alpha_{34}], [\alpha_{16} - \alpha_{35}]\}. \end{aligned}$$

In particular, by 5.3, a 2-form ω on M is symplectic if and only if

$$[\omega] = A[\alpha_{14}] + B[\alpha_{15} + \alpha_{24}] + C[\alpha_{26} - \alpha_{34}] + D[\alpha_{16} - \alpha_{35}], \tag{5.3}$$

where $ACD - B(C^2 + D^2) \neq 0$, $A, B, C, D \in \mathbb{R}$.

Claim 5.5 *Let ω be a symplectic form on M. Then the following holds:*

(i) *if $C^2 \neq D^2$, then $h_4 = 4$;*

(ii) *if $C^2 = D^2$ and $A^2 \neq 4B^2$, then $h_4 = 3$;*

(iii) *if $C^2 = D^2$ and $A^2 = 4B^2$, then $h_4 = 2$.*

Furthermore, $h_5 = 0$ for every symplectic form ω on M.

Proof : It follows from the Poincaré duality that $H^4(M) = \mathbb{R}^4$. Hence, by (5.2)

$$H^4(M) = \{[\alpha_{1246}], [\alpha_{1256}], [\alpha_{1356}], [\alpha_{1346} + \alpha_{2356}]\}$$

since the four cohomology classes from above are linearly independent. Furthermore, for every ω the image of the mapping $L : H^2(M) \longrightarrow H^4(M)$ is

$$\begin{aligned} \operatorname{Im} L &= \{-C[\alpha_{1246}] + D[\alpha_{1256}], 2D[\alpha_{1246}] - 2C[\alpha_{1256}], \\ &\quad -A[\alpha_{1246}] - 2B[\alpha_{1256}] - 2C[\alpha_{1356}] - D[\alpha_{1346} + \alpha_{2356}], \\ &\quad 2B[\alpha_{1246}] + A[\alpha_{1256}] - 2D[\alpha_{1356}] - C[\alpha_{1346} + \alpha_{2356}]\}, \end{aligned}$$

which has dimension 4 for $C^2 \neq D^2$, dimension 3 for $C^2 = D^2$ and $A^2 \neq 4B^2$, and dimension 2 for $C^2 = D^2$ and $A^2 = 4B^2$. The result follows from Corollary 3.4. $\boxed{QED}$

Now, the proof of the proposition follows from 2.11. $\boxed{QED}$

Proposition 5.6 *The compact nilmanifold of the type* $(0, 0, 0, 12, 14, 15 + 23 + 24)$ *is flexible.*

Proof : According to our assumption about the type of M, there exists a basis $\{\alpha_i\}_{i=1}^6$ of homogeneous 1-forms on M such that

$$d\alpha_1 = d\alpha_2 = d\alpha_3 = 0, \quad d\alpha_4 = \alpha_{12}, \quad d\alpha_5 = \alpha_{14}, \quad d\alpha_6 = \alpha_{15} + \alpha_{23} + \alpha_{24}.$$

The cohomology groups of degrees 1, 2 are:

$$\begin{aligned} H^1(M) &= \{[\alpha_1], [\alpha_2], [\alpha_3]\}, \\ H^2(M) &= \{[\alpha_{13}], [\alpha_{15}], [\alpha_{23}], [\alpha_{16} + \alpha_{25} - \alpha_{34}], [\alpha_{26} - \alpha_{45}]\}. \end{aligned}$$

In particular, if ω is a symplectic form on M then

$$[\omega] = A[\alpha_{13}] + B[\alpha_{15}] + C[\alpha_{23}] + D[\alpha_{16} + \alpha_{25} - \alpha_{34}] + E[\alpha_{26} - \alpha_{45}], \tag{5.4}$$

where $AE^2 + BDE - CDE - D^3 \neq 0$.

The following claim can be proved similarly to 5.5.

Claim 5.7 *Let ω be a symplectic form on M. Then the following holds:*

(i) *if $E \neq 0$, then $h_4 = 4$, $h_5 = 2$;*

(ii) *if $E = 0$, then $h_4 = 3$, $h_5 = 0$.* $\square$

Now, by 2.11, M is flexible. $\boxed{QED}$

We hope that now it is clear how to run over all the three remaining cases. So, below we omit the details while indicate the main steps of the corresponding calculations.

Proposition 5.8 *The compact nilmanifolds of types*

$(0,0,0,12,13+14,24)$, $(0,0,0,12,13,23)$ *and* $(0,0,0,0,12,13)$

are flexible.

Proof : **Case** $(0,0,0,12,13+14,24)$**:**

$$H^2(M) = \{[\alpha_{13}], [\alpha_{15}], [\alpha_{23}], [\alpha_{16}+\alpha_{25}+\alpha_{34}], [\alpha_{26}]\}, \tag{5.5}$$

and the 2-form ω on M is symplectic if and only if

$$[\omega] = A[\alpha_{13}] + B[\alpha_{15}] + C[\alpha_{23}] + D[\alpha_{16}+\alpha_{25}+\alpha_{34}] + E[\alpha_{26}],$$

where $D(BE-D^2) \neq 0$. Furthermore, if $EB+3D^2 \neq 0$ then $h_4 = 3$; otherwise, $h_4 = 2$. Finally, $h_5 = 0$ for every ω.

Case $(0,0,0,12,13,23)$**:**

$$H^2(M) = \{[\alpha_{14}], [\alpha_{15}], [\alpha_{16}+\alpha_{25}], [\alpha_{16}-\alpha_{34}], [\alpha_{24}], [\alpha_{26}], [\alpha_{35}], [\alpha_{36}]\},$$

and the 2-form ω on M is symplectic if and only if

$$[\omega] = A[\alpha_{14}] + B[\alpha_{15}] + C[\alpha_{16}+\alpha_{25}] + D[\alpha_{16}-\alpha_{34}] + E[\alpha_{24}] + F[\alpha_{26}] + G[\alpha_{35}] + H[\alpha_{36}],$$

where

$$ACH - AFG - BDF - BEH + DC^2 + CEG + CD^2 + DEG \neq 0.$$

Furthermore, if $C^2+CD+D^2-BF-EG+AH \neq 0$, then $h_4 = 8$; otherwise, $h_4 = 7$. Finally, $h_5 = 0$ for every ω.

Case $(0,0,0,0,12,13)$**:**

The second de Rham cohomology group is given by

$$H^2(M) = \{[\alpha_{14}], [\alpha_{15}], [\alpha_{16}], [\alpha_{23}], [\alpha_{24}], [\alpha_{25}], [\alpha_{34}], [\alpha_{26}+\alpha_{35}], [\alpha_{36}]\}.$$

The 2-form ω on M is symplectic if and only if

$$\begin{aligned}[\omega] = A[\alpha_{14}] &+ B[\alpha_{15}] + C[\alpha_{16}] + D[\alpha_{23}] + E[\alpha_{24}] + F[\alpha_{25}] \\ &+ G[\alpha_{34}] + H[\alpha_{26}+\alpha_{35}] + I[\alpha_{36}],\end{aligned}$$

where $-AFI + H^2A + BEI - BGH - CEH + CFG \neq 0$. Furthermore, if $H^2 - FI \neq 0$, then $h_4 = 8$; otherwise, $h_4 = 7$. Finally, $h_5 = 2$ for every ω.

Notice that in this last case the nilpotent Lie algebra (so the compact nilmanifold) is reducible of type $(1+5)$. $\boxed{QED}$

Comment 5.9 Here we want to say more about flexibility of manifolds appeared in 5.4 and 5.6.

(a) Consider the manifold from 5.4 and the family

$$\omega_t = -\frac{t}{2}(t-3)\alpha_{14} + \frac{1}{4}(t^2-5t+4)(\alpha_{15}+\alpha_{24}) - \frac{t}{2}(t-3)(\alpha_{26}-\alpha_{34}) + \alpha_{16} - \alpha_{35}$$

of closed 2-forms on M. The form ω_t is symplectic if

$$t^6 - 11t^5 + 39t^4 - 45t^3 + 4t^2 - 20t + 16 \neq 0.$$

This polynomial has two real roots, and both of them lie out of the interval $(1-\varepsilon, 4)$. So, because of 5.5

$$h_4(\omega_2) = 2, \quad h_4(\omega_1) = h_4(\omega_{\frac{3+\sqrt{17}}{2}}) = 3,$$

and $h_4(\omega_t) = 4$ for all other $t \in (1-\varepsilon, 4)$.

(b) Consider the manifold from 5.6 and the family

$$\omega_t = (1-t)\alpha_{13} - t(\alpha_{16} + \alpha_{25} - \alpha_{34}) + (1-t)(\alpha_{26} - \alpha_{45})$$

of closed 2-forms. Since the polynomial $AE^2 + BDE - CDE - D^3 = 3t^2 - 3t + 1$ has no real roots, we conclude that ω_t is a family of symplectic structures and

(i) $h_4(\omega_1) = 3$, $h_5(\omega_1) = 0$;

(ii) $h_4(\omega_t) = 4$, $h_5(\omega_t) = 2$ for $t \neq 1$.

Remark 5.10 Minding Theorem 5.1, it is natural to ask whether there exist flexible nilmanifolds of dimension greater than 6. Taking into account the results of the next section, we see that the answer is affirmative. However, the question remains open for irreducible compact nilmanifolds of dimension greater than 6.

Remark 5.11 We have also considered the 6-dimensional compact completely solvable manifold M constructed by Fernández–de León–Saralegi [6]. This compact manifold does not satisfy the Hard Lefschetz condition (although it is of Lefschetz type). We have

$$b_1(M) = b_5(M) = 2,\ b_2(M) = b_4(M) = 3,\ b_3(M) = 4.$$

Furthermore, $h_3(M,\omega) = 4$ and $h_4(M,\omega) = h_5(M,\omega) = 2$ for every symplectic form ω on M. We do not explain the details because M is not flexible.

The following question also seems to be interesting.

Question: Is there a Nomizu's type result for compact nilmanifolds (more generally, homogeneous spaces) and the symplectically harmonic cohomology $H^*_{\mathrm{hr}}(M)$. In another words, does a symplectically harmonic de Rham cohomology class contain a homogeneous symplectically harmonic representative (if we are considering homogeneous symplectic structures)?

The answer is affirmative for degrees $k \leq 2$ and $k \geq 2m-2$. Indeed, let G/Γ be a $2m$-dimensional compact nilmanifold with a homogeneous symplectic form ω, and let $\mathfrak{g}$ be the Lie algebra of G. Since the image of a homogeneous form under each of the operators $*$, d and L is homogeneous, the (finite dimensional) subspaces $\Omega^*_{\mathrm{hr}}(\mathfrak{g}^*)$ and $\Lambda^*(\mathfrak{g}^*)$ are $\mathfrak{sl}(2)$-submodules of $\Omega^*(G/\Gamma)$. Therefore, obvious analogs of Proposition 2.5, Corollaries 3.2 and 3.4 hold for $\mathfrak{g}^*$ and the result follows from the Nomizu's theorem.

6 Product formula for symplectically harmonic cohomology

Let (M^{2m}, ω_1) and (N^{2n}, ω_2) be two symplectic manifolds. Consider the symplectic product manifold $(M \times N, \omega)$ where $\omega = p_1^{\#}\omega_1 + p_2^{\#}\omega_2$ and

$$p_1 : M \times N \to M, \quad p_2 : M \times N \to N$$

are the projections. Given two forms $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(N)$, consider the form

$$\alpha \boxtimes \beta := (p_1^{\#}\alpha) \wedge (p_2^{\#}\beta) \in \Omega^{p+q}(M \times N).$$

Proposition 6.1 ([2])

$$*(\alpha \boxtimes \beta) = (-1)^{pq}(*_1\alpha) \boxtimes (*_2\beta),$$

□

Corollary 6.2 (i) $i(\Pi)(\alpha \boxtimes \beta) = (i(\Pi_1)\alpha) \boxtimes \beta + \alpha \boxtimes (i(\Pi_2)\beta)$*;*

(ii) $\delta(\alpha \boxtimes \beta) = (\delta_1\alpha) \boxtimes \beta + (-1)^p \alpha \boxtimes (\delta_2\beta)$*;*

(iii) $\Omega^p_{\mathrm{hr}}(M) \boxtimes \Omega^q_{\mathrm{hr}}(N) \subset \Omega^{p+q}_{\mathrm{hr}}(M \times N)$*;*

(iv) *for all k we have*

$$\sum_{p+q=k} h_p(M)h_q(N) \leq h_k(M \times N).$$

□

Question: When the inequality in 6.2(iv) turns out to be the equality?

Now we consider the Lefschetz map $L_{[\omega]}^{m+n-k} : H^k(M \times N) \longrightarrow H^{2m+2n-k}(M \times N)$.

Proposition 6.3

$$\begin{aligned} h_{2(m+n)-1}(M \times N) &= h_{2m-1}(M) + h_{2n-1}(N), \\ h_{2(m+n)-2}(M \times N) &= h_{2m-2}(M) + h_{2m-1}(M)h_{2n-1}(N) + h_{2n-2}(N). \end{aligned}$$

Proof : Because of the Künneth isomorphism $H^*(M \times N) \cong H^*(M) \otimes H^*(N)$, we conclude that

$$\operatorname{Im}\left(L_{[\omega]}^{m+n-1}\right) = [\omega_1]^m \otimes \left(\operatorname{Im} L_{[\omega_2]}^{n-1}\right) \oplus \left(\operatorname{Im} L_{[\omega_1]}^{m-1}\right) \otimes [\omega_2]^n,$$

and the first equality follows from 3.4. Similarly,

$$\begin{aligned} \operatorname{Im}\left(L_{[\omega]}^{m+n-2}\right) &= \{([\omega_1]^{m-1}u) \otimes [\omega_2]^{n-1} + ([\omega_1]^{m-2}u) \otimes [\omega_2]^n \mid u \in H^2(M)\} \\ &\oplus \{[\omega_1]^m \otimes ([\omega_2]^{n-2}v) + [\omega_1]^{m-1} \otimes ([\omega_2]^{n-1}v) \mid v \in H^2(N)\} \\ &\oplus \{([\omega_1]^{m-1}w_1) \otimes ([\omega_2]^{n-1}w_2) \mid w_1 \in H^2(M),\, w_2 \in H^2(N)\}. \end{aligned}$$

Now, in view of 3.4, the computation of dimensions completes the proof. $\boxed{QED}$

Corollary 6.4 *Let M^{2m} be a manifold which admits a family of symplectic forms such that the symplectically harmonic Betti number h_{2m-k} varies for $k = 1$ or $k = 2$. Then $M \times N$ is a flexible manifold whenever a manifold N admits a symplectic structure.*

Proof : If $h_{2m-1}(M)$ varies then the result follows from the first equality of 6.3. If $h_{2m-2}(M)$ varies but $h_{2m-1}(M)$ does not vary then the result follows from the second equality of 6.3. $\boxed{QED}$

7 Duality

Consider a symplectic manifold (M^{2m}, ω) and the chain complex

$$\cdots \longrightarrow \Omega^{k+1}(M) \xrightarrow{\delta} \Omega^k(M) \xrightarrow{\delta} \Omega^{k-1}(M) \longrightarrow \cdots$$

with δ as in (2.1). The following proposition follows directly from the definition of δ.

Proposition 7.1 (i) $\delta\alpha = 0$ *if and only if* $d * \alpha = 0$;

(ii) $\alpha \in \operatorname{Im}\delta$ *if and only if* $*\alpha \in \operatorname{Im} d$. $\square$

We define

$$H_\delta^k(M) = H_\delta^k(M,\omega) = \operatorname{Ker}\delta^k / \operatorname{Im}\delta^{k+1} \text{ where } \delta^i = \delta : \Omega^i \to \Omega^{i-1}.$$

Corollary 7.2 *The operator $* : \Omega^k \to \Omega^{2m-k}$ induces an isomorphism*

$$* : H^k(M) \to H_\delta^{2m-k}(M).$$

In particular, $H_\delta^k(M) = H^k(M)$ for M closed. □

We dualize the definition of symplectically harmonic Betti numbers h_k by setting

$$h_k^*(M) = h_k^*(M,\omega) := \dim \left(\Omega_{\mathrm{hr}}^k / \operatorname{Im}\delta \cap \Omega_{\mathrm{hr}}^k\right)$$

Corollary 7.3 *$h_{m-k}^*(M) = h_{m+k}(M)$.* □

In particular, in view of 3.2, if M is closed then $h_{2m-k}^*(M) = b_{2m-k}(M)$ for $k = 0, 1, 2$.

It is clear that many other results of Sections 2 and 3 can be dualized in a similar way. We leave it to the reader.

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